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This report represents results obtained at the Courant Institute of Mathematical Sciences, New York University, and the Research Center, Rockland State Hospital, under respective Public Health Service Grants No. GM 16202-01 (National Institute of General Medical Sciences) and No. MH 14934-02 (National Institute of Mental Health).

Abstract

Suppose that $X_{(1)} < \ldots < X_{(n)}$ are the order statistics of a sample of size n from a continuous and symmetric distribution with a location parameter θ.

We consider a (fictitious) subsample of size k drawn randomly from the order statistics of size n, and let $Y_{(1)} < \ldots < Y_{(k)}$ be the order statistics obtained from the subsample.

We shall determine the coefficients of a linear combination

$$\sum c_\alpha Y_{(\alpha)}$$

so as to minimize $V(\sum c_\alpha Y_{(\alpha)} \mid X_{(1)} < \ldots < X_{(n)})$ under the condition that $\sum c_\alpha = 1$. Then we take the conditional expectation and regard it as an estimator of θ,

$$\begin{aligned}\hat{\theta}^* &= E(\sum c_\alpha Y_{(\alpha)} \mid X_{(1)} < \ldots < X_{(n)}) \\ &= \sum c_\alpha T_\alpha\end{aligned}$$

where, actually, $T_\alpha = E(Y_{(\alpha)} \mid X_{(1)} < \ldots < X_{(n)})$ is a linear combination of the $X_{(i)}$'s. For a wide class of distributions satisfying a certain set of regularity conditions it is shown that $\hat{\theta}$ is asymptotically efficient, when $n \to \infty$ and simultaneously, $k \to \infty$ in an appropriate way.

Monte Carlo experimentation shows that this estimator, or rather a modified version with symmetric coefficients i.e. $c_\alpha = c_{k-\alpha+1}$, attains high relative efficiency for several

varieties of distributions even for small samples of size $n = 10, 15, 20$ or even $n = 5$.

We can also find a natural way of "Studentization" and asymptotic confidence intervals are obtained. Utilizing Monte Carlo techniques, these intervals also seem to be satisfactory for small samples.

§1. Introduction.

Most of the so-called "robust" estimators for location parameters are either linear combinations of order statistics or closely related to them. Trimmed and Winsorized means (Tukey [28]) are the most extensively studied examples (Bickel [4], Gastwirth and Cohen [12], Crow and Siddiqui [10]). Also, minimax linear unbiased estimators have been discussed (Birnbaum et al. [5,6], Gastwirth and Rubin [13], Yhap [29]). Huber's estimator (17] is primarily of another kind, but is reduced to a kind of trimmed mean for his specific choice of his ρ function. Hodges-Lehmann's [16] is also of another kind, and seems to be the only practically tractable estimator derived from the rank tests. (For general estimators based on rank tests see Adichie [1].) But it has also been shown that it is also closely related to a linear combination of order statistics.

It has been known since Bennett [2], Jung [19] and Blom [7] that the best linear unbiased estimator (BLUE) is asymptotically efficient in the sense that the variance of the BLUE is asymptotically equal to the Cramer-Rao bound for the variances of the unbiased estimators (Jung [19]), or asymptotic variance of the BLUE is equal to it (e.g. Chernoff et al. [8]). Of course, in order to calculate the coefficient of BLUE, it is necessary to compute the variance-covariance matrix of order statistics, but Blom [7] also showed that asymptotically the coefficients $c(i|n)$ of the

BLUE of the i-th order statistic in a sample of size n, can be expressed

$$c(i|n) \propto -\frac{\partial^2}{\partial x^2} \log f(x)\Big|_{x=\xi(\frac{i}{n+1})} \qquad (1.1)$$

where f denotes the density and $\xi(\ )$ the quantile of the distribution. This expression has suggested that if we have a very large sample, we might be able to estimate the density frunction from the sample, and then construct an estimator which is asymptotically equivalent to the BLUE without assuming knowledge of the shape of the distribution. Stein [22] and Hajek [15] investigated such possibilities (also Bhattacharya [3]). However, since in the above formula the second order derivative of the density is involved, it is intuitively obvious that the sample size must be prohibitively large in order that the asymptotic efficiency be attained.

There are many other possible ways of using the information contained in the sample configuration to choose a linear estimator so as to attain high or full asymptotic efficiency for a wide class of shapes of the distribution. Such estimators may be called <u>quasilinear</u> estimators; one example was given by Hogg [17].

In this paper we shall introduce a quasilinear estimator which is shown to be asymptotically (nearly) efficient for a wide class of symmetric distributions with certain regularity conditions. Monte Carlo experiments

show that it has quite high relative efficiencies even for moderate sample sizes such as n = 10, 15, 20. Further, there is a natural method of "Studentization" for this estimator, which is shown to be asymptotically valid for any (regular) shapes of distributions, and Monte Carlo results also indicate that for small samples the resulting confidence intervals are uniformly approximately valid, and more or less biased on the conservative side when the underlying distribution is long-tailed.

The basic idea of the method is as follows. Although it is quite impossible to estimate the variance-covariance matrix of the order statistics directly from the sample, it is possible to estimate the variance-covariance of the order statistics of a sample of size k, from a sample of size n, and if n is appreciably larger than k, the estimates would be accurate. But if the distribution is regular, this information about the variance-covariance matrix of the order statistics of a sample of size k can be effectively used to construct a "nearly" efficient estimator based on the sample of size n provided that k is also large.

This method can be regarded as a refinement of the method proposed by the author [24] which was based on random division of the samples into several sets of subsamples, and also asymptotically uniformly efficient, but which had an undesirable property of not being a function of the order statistics.

§2. The Method.

Suppose that $X_1 \ldots X_n$ are independently and identically distributed random variables with continuous distribution with a location parameter θ. We shall denote the density function by $f(x-\theta)$, and we shall assume the following:

(1) $f(x) > 0$ for all x

(2) $f(-x) = f(x)$, that is the distribution is symmetric

(3) $f(x)$ is continuous and almost everywhere differentiable, and

$$I_f = \int \frac{\{f'(x)\}^2}{f(x)} \, dx < \infty \tag{2.1}$$

Then, under some well known regularity conditions, it is known that (Cf. Cramer [9]) for any unbiased estimator $\hat{\theta}$ of θ,

$$V(\hat{\theta}) \geq \frac{1}{n \, I_f} \quad . \tag{2.2}$$

Now let $X_{(1)} < \ldots < X_{(n)}$ be the order statistics obtained by rearranging $X_1 \ldots X_n$. Consider a linear combination

$$\hat{\theta} = \sum_{i=1}^{n} c_i \, X_{(i)} \; . \tag{2.3}$$

If $c_i = c_{n-i+1}$ for $i = 1,2,\ldots$ and $\sum c_i = 1$ in (2.3), $\hat{\theta}$ is unbiased for θ for any symmetric distribution, and we shall call it a linear unbiased estimator.

For any fixed symmetric f, let c_i^* be defined as the value of the coefficient minimizing

$$V(\hat{\theta}) = V(\sum c_i \, X_{(i)}) \; ,$$

under the condition that $\sum c_i = 1$. Then it is well known (Lloyd [20]) that c_i is given by

$$c_i^* = \frac{\sum_j \sigma^{ij}}{\sum_i \sum_j \sigma^{ij}} \tag{2.4}$$

where $\{\sigma^{ij}\} = \{\sigma_{ij}\}^{-1}$ and $\sigma_{ij} = E(X_{(i)}X_{(j)}) - E(X_{(i)})E(X_{(j)})$. It is easily shown that $c_{n-i+1}^* = c_i^*$, hence

$$\hat{\theta}_f^* = \sum c_i^* X_{(i)} \tag{2.5}$$

is a linear unbiased estimator, and is of minimum variance when f is the time distribution, and is called the best linear unbiased estimator (for short BLUE).

It was proved by Bennett [2], Jung [19] and Blom [7] that under some regularity conditions BLUE is asymptotically efficient, in the sense that

$$nV(\hat{\theta}_f^*) \longrightarrow \frac{1}{I_f} \tag{2.6}$$

More precisely, for some class $\mathcal{F}$ of symmetric distributions satisfying some regularity conditions, the convergence in (2.6) is uniform, that is

$$nV(\hat{\theta}_f^*) = \frac{1}{I_f} + O(\frac{1}{n}) \tag{2.7}$$

or still more precisely, for some $K > 0$

$$nV(\hat{\theta}_f^*) \leq \frac{1}{I_f} + \frac{K}{n} \qquad \text{for all f in } \mathcal{F}. \tag{2.8}$$

Hence for any $\varepsilon > 0$, if we take k large enough, we have

$$kV(\hat{\theta}_f^*) \leq \frac{1}{I_f} + \varepsilon \qquad \text{for all } f \text{ in } \mathcal{F} . \tag{2.9}$$

Now suppose that k is fixed but large enough to satisfy (2.9). We shall denote the BLUE for the sample of size k specifically by

$$\hat{\theta}_f^*(k) = \sum_{i=1}^{k} c_i^* X_{(i)} . \tag{2.10}$$

Now consider the order statistic of size n, $n > k$, $X_{(1)} < \ldots < X_{(n)}$, and suppose (hypothetically) that a subsample of size k be randomly chosen from this sample, the order statistic obtained from which is denoted by $Y_{(1)} < \ldots < Y_{(k)}$. Then

$$Y_{(\alpha)} = X_{(i_\alpha)} , \quad \alpha = 1, \ldots k, \; i_1 < i_2 < \ldots < i_k \tag{2.11}$$

The joint distribution of $Y_{(1)} < \ldots Y_{(k)}$ is exactly the same as the joint distribution of the order statistics of size k from the original distribution. Hence if we put

$$\hat{\hat{\theta}}_f = \sum_{j=1}^{k} c_\alpha^* Y_{(\alpha)} \tag{2.12}$$

we have $kV(\hat{\hat{\theta}}_f) = kV(\hat{\theta}_f^*) \leq \frac{1}{I_f} + \varepsilon$.

Now we shall denote the conditional expectation of θ given the order statistics $X_{(1)} < \ldots < X_{(n)}$ by

$$\hat{\theta}_f^0 = E(\hat{\hat{\theta}}_f \mid O_n) = \sum_{j=1}^{k} c_\alpha^* T_\alpha \tag{2.13}$$

where $T_\alpha = E(Y_{(\alpha)} | O_n)$, and O_n is the abbreviation denoting the order statistic of size n. T_α can be expressed as

$$T_\alpha = \sum_{i=1}^{n} P_\alpha^i \, X_{(i)} \tag{2.14}$$

and

$$P_\alpha^i = P_r\{Y_{(\alpha)} = X_{(i)}\} = {}_{i-1}C_{\alpha-1} \; {}_{n-i}C_{k-\alpha} \; / \; {}_nC_k \quad \text{for } \alpha \leq i \leq n-k+\alpha \tag{2.15}$$

$$= 0 \quad \text{otherwise}$$

Here we need the following lemma.

LEMMA. Suppose that $\Phi(x_1 \ldots x_k)$ is a real valued function of k real variables. Let $O_n = \{X_{(1)} < \ldots < X_{(n)}\}$ be the order statistics of a sample of size n (> k) from a continuous distribution. Let us denote

$$\psi(X_{(1)} \ldots X_{(n)}) = E\{\Phi(X_1 \ldots X_k) \mid O_n\} \tag{2.16}$$

then $V(\psi) \leq kV(\Phi)/n$ provided that $V(\Phi)$ is finite.

<u>Proof</u>: Since

$$\psi = \frac{1}{{}_nC_k} \sum_{i_1 < \ldots i_k} \Phi(X_{(i_1)} \ldots X_{(i_k)})$$

the lemma is directly derived from the following:

If $X_1, X_2, \ldots$ are independently and identically distributed, then

$$\text{cov}\,\{\Phi(X_1 \ldots X_j, X_{j+1} \ldots X_k), \Phi(X_1 \ldots X_j, X_{k+1} \ldots X_{2k-j})\}$$

$$\leq \frac{j}{k} V(\Phi) \; .$$

In order to prove the above statement we can assume with no loss of generality that $V(\Phi) = 1$.

Define

$$c_j = \operatorname{cov}\{\Phi(X_1 \ldots X_j\, X_{j+1} \ldots X_k)\, \Phi(X_1 \ldots X_j\, X_{k+1} \ldots X_{2k-j})\}$$

$$j = 0, \ldots, k.$$

Suppose that $X_1 \ldots X_j$, $X_{11} \ldots X_{1\ell}$, $\ldots X_{h1} \ldots X_{kh}$, $h = k-j$, are independently and identically distributed random variables.

$$V\{\sum_{i=1}^{h} \Phi(X_1 \ldots X_j, X_{1i} \ldots X_{hi}) - \sum_{i=1}^{h} \Phi(X_1 \ldots X_j\, X_{i'1} \ldots X_{i'n})\} \geq 0$$

implies that

$$2h + 2h(h-1)c_j - 2h^2 c_{j+1} \geq 0$$

$$c_{j+1} \leq \frac{1}{k-j}\{1 + (k-j-1)c_j\}, \quad j = 0,1,2,\ldots k-1. \tag{2.17}$$

Since $c_0 = 0$, it is proved by induction that $c_j \leq j/k$, $j = 1,\ldots,k$. Q.E.D.

From the lemma we have

$$V(\hat{\theta}_f^0) \leq \frac{k}{n} V(\hat{\hat{\theta}}_f)$$

hence

$$nV(\hat{\theta}_f^0) \leq \frac{1}{I_f} + \varepsilon . \tag{2.18}$$

Thus it was shown that $\hat{\theta}_f^0$ is ε-efficient for $n > k$ for all f in $\mathcal{F}$.

The coefficients c_α^* in $\hat{\theta}_f^0$ given by (2.4) are

dependent on the assumed density f, but if n is larger than k it can be also estimated from the sample. Define

$$s_{\alpha\beta} = V(Y_{(\alpha)} \mid O_n) \qquad \text{for } \alpha = \beta$$

$$= \text{cov}\,(Y_{(\alpha)}, Y_{(\beta)} \mid O_n) \qquad \alpha \neq \beta$$

where V(|) and Cov (|) denote the conditional variance and covariance given the order statistics. They are given by

$$\begin{aligned} s_{\alpha\alpha} &= E(Y^2_{(\alpha)} \mid O_n) - T^2_\alpha \\ &= \sum P^i_\alpha X^2_{(i)} - T^2_\alpha \\ s_{\alpha\beta} &= E(Y_{(\alpha)}, Y_{(\beta)} \mid O_n) - T_\alpha T_\beta \\ &= \sum_{i<j}\sum P^{ij}_{\alpha\beta} X_{(i)} X_{(j)} - T_\alpha T_\beta \quad \text{for } \alpha < \beta \end{aligned} \tag{2.19}$$

$$s_{\beta\alpha} = s_{\alpha\beta}$$

where

$$P^{ij}_{\alpha\beta} = P_r\{Y_{(\alpha)} = X_{(i)},\ Y_{(\beta)} = X_{(j)}\} = \frac{{}_{i-1}C_{\alpha-1}\ {}_{j-i-1}C_{\beta-\alpha-1}\ {}_{n-j}C_{k-\beta}}{{}_nC_k}$$

$$\text{for } \alpha \leq i,\ \beta \leq n-j+k$$

$$= 0 \qquad \text{otherwise} \tag{2.20}$$

And if the distribution has the finite fourth order moment,

$$V(Y_{(\alpha)}\ Y_{(\beta)}) = \tau_{\alpha\beta} < \infty$$

and from the lemma above,

$$V(s_{\alpha\beta}) \leq \frac{k}{n}\tau_{\alpha\beta}\ . \tag{2.21}$$

Hence for fixed κ,

$$s_{\alpha\beta} \longrightarrow \sigma_{\alpha\beta} \quad \text{in probability} \tag{7.22}$$

as $n \to \infty$. Hence if we define

$$c_\alpha = \sum s^{\alpha\beta} \Big/ \sum\sum s^{\alpha\beta}$$

$$\text{where} \quad \{s^{\alpha\beta}\} = \{s_{\alpha\beta}\}^{-1} \ ;$$

then $c_\alpha \to c_\alpha^*$ in probability, and if we put

$$\hat{\theta}^* = \sum c_\alpha T_\alpha$$

$\hat{\theta}^*$ and $\hat{\theta}_f^0$ are asymptotically equivalent, or more precisely,

$$\sqrt{n}\,(\hat{\theta}^* - \hat{\theta}_f^0) = \sum_\alpha (c_\alpha - c_\alpha^*)\sqrt{n}\,T_\alpha \longrightarrow 0$$

in probability. Thus $\hat{\theta}^*$ is asymptotically equivalent to an ε-efficient estimator $\hat{\theta}_f^0$ for any f in $\mathcal{F}$.

If $\mathcal{F}$ is properly defined, we can define a bounded open subset C of k dimensional Euclidean space such that for any f in $\mathcal{F}$ the corresponding vector $\{c_\alpha^*\}$ lies in C. Then we can define

$$\hat{c}_\alpha = c_\alpha \qquad \text{if} \quad \{c_\alpha\} \in C \ ;$$

otherwise we determine $\hat{c}_\alpha$ so as to minimize

$$\sum\sum s_{\alpha\beta}\,\hat{c}_\alpha\,\hat{c}_\beta$$

under the condition $\sum \hat{c}_\alpha = 1$ and $\{\hat{c}_\alpha\} \in \bar{C}$ where $\bar{C}$ denotes the convex closure of C.

Then it can be shown that

$$E(\sum (\hat{c}_\alpha - c^*_\alpha)^2) \longrightarrow 0 \quad \text{and} \quad \sum (\hat{c}_\alpha - \hat{c}^*_\alpha)^2 \text{ is bounded;}$$

therefore if we define

$$\hat{\hat{\theta}}^* = \sum \hat{c}_\alpha T_\alpha , \tag{2.23}$$

$$nE(\hat{\hat{\theta}}^* - \hat{\theta}^0_f)^2 \leq E[(\sum (\hat{c}_\alpha - c^*_\alpha)^2)(\sum n T^2_\alpha)]$$

$$\longrightarrow 0$$

as $n \longrightarrow \infty$. Hence

$$nV(\hat{\hat{\theta}}^*) \leq \frac{1}{I_f} + \varepsilon' \tag{2.24}$$

when n is large enough for any $\varepsilon' > \varepsilon$.

Thus if the class $\mathcal{F}$ satisfies a set of regularity conditions thus far mentioned, for any $\varepsilon > 0$ we can find a k for which (2.9) is satisfied with $\varepsilon/2$, for all f in $\mathcal{F}$, and then if we take n large enough, (2.24) is satisfied with $\varepsilon' = \varepsilon$, for all f in $\mathcal{F}$.

Finally, if we choose a sequence of positive numbers $\varepsilon_m \downarrow 0$, and construct a sequence of estimators $\hat{\hat{\theta}}^*_m$ in the way described above with $\varepsilon = \varepsilon_m$, then we have

$$\lim_{m \to \infty} n_m V(\hat{\hat{\theta}}^*_m) \longrightarrow \frac{1}{I_f} , \tag{2.25}$$

where $\{n_m\}$ is a sequence of integers tending to infinity.

Also as was shown by Chernoff et al. [8], $\sqrt{n}\,(\theta^0_f - \theta)$ is asymptotically normal $N(0\ 1/I_f)$ when $k \longrightarrow \infty$; hence if the distributions in $\mathcal{F}$ satisfy the conditions mentioned

in [8] uniformly, then if we take k large enough, we can obtain

$$\sup_t \left| P_r\{ \sqrt{n\, I_f}\, (\hat{\theta}_f^0 - \theta) < t\} - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \right| < \varepsilon ,$$

for arbitrary $\varepsilon > 0$ for all f in $\mathcal{F}$.

Then again if we take n large enough, we have

$$\sup_t \left| P_r\{ \sqrt{n\, I_f}\, (\hat{\theta}^* - \theta) < t\} - \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \right| < 2\varepsilon ,$$

for all f in $\mathcal{F}$. Hence when n is large enough, the difference between the distribution of θ^* and that of the normal becomes smaller than any prescribed number, uniformly for all the distributions in $\mathcal{F}$.

Assuming that the distribution is unknown, the variance of $\hat{\theta}^*$ is also unknown, but is estimated asymptotically in the following way.

When k is large,

$$V(\hat{\hat{\theta}}_f) \sim \frac{1}{kI}$$

and

$$V(\hat{\theta}_f^0) \sim \frac{1}{nI} .$$

But

$$\begin{aligned} V(\hat{\hat{\theta}}_f) &= E(V(\hat{\hat{\theta}}_f \mid O_n)) + V(E(\hat{\hat{\theta}}_f \mid O_n)) \\ &= E(V(\hat{\hat{\theta}}_f \mid O_n)) + V(\hat{\theta}_f^0) . \end{aligned}$$

Hence

$$\frac{k}{n-k} E(V(\hat{\hat{\theta}}_f \mid O_n)) \sim V(\hat{\theta}_f^0) \tag{2.26}$$

Since $V(\hat{\hat{\theta}}_f \mid O_n) = \sum\sum c^*_\alpha c^*_\beta s_{\alpha\beta}$, if we replace c^*_α by c_α we have an estimator for $V(\hat{\theta}^0_f)$ given by

$$\hat{V}(\hat{\theta}^0_f) = \frac{k}{n-k} \frac{1}{\sum_\alpha \sum_\beta s^{\alpha\beta}} \tag{2.27}$$

And if we can assume asymptotic normality, we have the asymptotic confidence interval for θ.

We shall introduce a slight modification for $\hat{\theta}^*$. Since the distribution is assumed to be symmetric,

$$\sigma_{\alpha\beta} = \sigma_{k-\alpha+1,k-\beta+1}$$

Hence we can use $\tilde{s}_{\alpha\beta} = (s_{\alpha\beta} + s_{k-\alpha+1,k-\beta+1})/2$ instead of $s_{\alpha\beta}$. We shall denote by $\tilde{\theta}^*$ the estimator of the form

$$\tilde{\theta}^* = \sum \tilde{c}_\alpha T_\alpha$$

$$\text{where} \quad \tilde{c}_\alpha = \sum_\beta \tilde{s}^{\alpha\beta} / \sum_\alpha \sum_\beta \tilde{s}^{\alpha\beta} , \quad \{\tilde{s}^{\alpha\beta}\} = \{\tilde{s}_{\alpha\beta}\}^{-1}. \tag{2.28}$$

§5. Monte Carlo Results.

We performed several sets of Monte Carlo experiments for the following set of distributions.

1. Normal

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

2. 50% Contaminated normal

$$f(x) = 0.95 \times \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + 0.05 \times \frac{1}{\sqrt{18\pi}} e^{-x^2/18}$$

3. 10% Contaminated normal

$$f(x) = 0.90 \times \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + 0.10 \times \frac{1}{\sqrt{18\pi}} e^{-x^2/18}$$

4. Logistic

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \frac{1}{1 + \exp\left(-\sqrt{\frac{3}{\pi}}\, x\right)}$$

5. Double exponential

$$f(x) = \frac{1}{2} e^{-|x|}$$

6. Tukey's [14] special distribution

$$X = \frac{1}{0.2} [U^{-0.1} - (1 - U)^{-0.1}]$$

where U is a uniform random variable over [0 1].

7. t-distribution with d.f. 2

$$f(x) = \frac{1}{(1 + x^2)^{3/2}}$$

8. Cauchy

$$f(x) = \frac{\sqrt{2}}{\pi} \frac{1}{1+2x^2}$$

9. Rectangular

$$f(x) = \frac{1}{\sqrt{12}} \quad , \quad |x| \leq \sqrt{3}$$

10. Triangular

$$f(x) = \sqrt{\frac{2}{3}} - \frac{2}{3}|x| \quad , \quad |x| \leq \sqrt{\frac{3}{2}}$$

11. Quadratic

$$f(x) = \frac{3}{8} - \frac{1}{8}x^2 \quad , \quad |x| \leq 1$$

$$= \frac{1}{16}(3-|x|)^2, \quad 1 \leq |x| \leq 3$$

$X = 2(U+V+W) - 3$. U, V and W are independent uniform random variables.

At the first stage we computed the estimates from random samples of size $n = 10$, from each of the populations listed above except t(d.f. 2), for $k = 3,4,5,6,7,8$. $N = 1000$ samples are drawn. Both symmetrized and unsymmetrized estimates are computed and we estimated the variances of estimators for each case. The results are listed in Table 1 and Table 2, where relative efficiencies are calculated by

$$\text{Eff}(\theta) = \frac{\text{Variance of BLUE}}{\text{Variance of } \hat{\theta}} .^{1/}$$

1. The variances and coefficients of BLUE's were given to me by Dr. Morris Meisner who obtained the numerical results in preparing for a joint paper by Birnbaum, Laska and Meisner [6].

The results showed a surprising superiority of the symmetrized estimator over the unsymmetrized ones, so that in the subsequent experiments, only symmetrized ones are used.

We also computed the means of

$$M = \frac{1}{\sum_{\alpha} \sum_{\beta} \tilde{s}_{\alpha\beta}}$$

and computed the ratio

$$\frac{\text{Mean of } M}{\text{Variance of } \hat{\theta}}$$

which is tabulated in Table 3. The values are fairly uniform between different shapes of distributions, except for Cauchy, and are uniformly smaller than the asymptotic value $(n-k)/k$.

As a rough approximation based on an intuitive conjecture, we put them approximately equal to

$$c_k = \frac{(n-k)(n-k+1)}{k(n-1)} . \qquad (3.1)$$

We computed studentized values using this approximation by

$$\frac{\hat{\theta}}{\sqrt{M/c_k}}$$

and computed cumulative distributions (Table 4). Again the distributions for different shapes are fairly uniform, also except for the case of Cauchy, in which the distribution is more concentrated than the other cases. And the distributions (except for Cauchy) of the studentized values are seen to be fairly close to t-distributions with degrees of freedom $n-k+1$.

Also for fixed $k = 6$, we computed the variances and relative efficiencies of estimators for $n = 10,15,20$, with $N = 1000$, the results being given in Table 5.

In view of the results obtained by the first set of Monte Carlo experiments, it seems to be certain that the relative efficiencies of our estimators are high, and this implies that the correlation between BLUE's and our estimator must be close to 1. This implies that we can estimate the difference

$$V(\hat{\theta}^*) - V(\hat{\theta}_{BLUE})$$

much more accurately thatn $V(\hat{\theta}^*)$ itself from the mean of

$$\hat{\theta}^{*2} - \hat{\theta}^2_{BLUE} \tag{3.2}$$

Since we have exact values of $V(\hat{\theta}_{BLUE})$ for several shapes of distributions, we can estimate $V(\hat{\theta}^*)$ by

$$\hat{V}(\hat{\theta}^*) = \text{Mean of } (\hat{\theta}^{*2} - \hat{\theta}^2_{BLUE}) + V(\hat{\theta}_{BLUE}) \ . \tag{3.3}$$

In the second stage of the Monte Carlo experiments, we used this "difference" estimator, and also computed a 95 percent confidence interval based on the sample variances of the difference (3.2) using normal approximation for the means. The results are given in Tables 6, 7 and 8 for $n = 10, 15, 20$, and for the cases where the BLUE could be obtained. One exceptional case is rectangular where instead of $\hat{\theta}_{BLUE} = (X_{min} + X_{max})/2$, $\bar{X}$ = sample mean is used (except for the case $k = 6$).

Also error frequencies of the confidence intervals based on the above approximations of the studentized values, and numbers of the cases among N = 2000 trials in which the true value falls outside the intervals with nominal confidence levels 99, 98, 95, 90 percent are tabulated in Tables 9, 10, and 11.

Finally, for k = 3 and k = 4, since the coefficients of BLUE's are all nonnegative except for extremely irregular distributions, and surely so for those shapes included in our list, we tried the estimator with nonnegative coefficients, that is we put the condition that $\hat{c}_\alpha \geq 0$. The results for this estimator are tabulated in Table 12 for n = 5 and 10.

Table 1. Variances and Relative Efficiencies of the Symmetrized Estimators. n = 10 N = 1000

k =	3	4	5	6	7	8	BLUE
Normal	.1056	.1076	.1109	.1137	.1164	.1187	.1000
	94%	92%	90%	87%	86%	84%	100%
CN .05	.1286	.1282	.1285	.1309	.1339	.1354	.1196
	93	94	94	92	90	89	100%
CN .10	.1555	.1540	.1507	.1524	.1544	.1563	.1358
	87	88	90	89	88	87	100%
Logist	.0964	.0979	.0991	.1016	.1034	.1045	.0934
	97	95	94	92	91	89	100%
Dl. Exp.	.1630	.1631	.1608	.1626	.1596	.1633	.1399
	86	86	87	86	88	86	100%
Spec.	.1053	.1060	.1054	.1078	.1087	.1099	.0957
	91	90	91	89	88	87	100%
Cauchy	1.1910	.8392	.2290	.2220	.2024	.2146	.1631
	14	19	72	74	81	76	100%
Rect.	.0947	.0905	.0935	.0953	.1050	.1095	.0455
	48	50	48	47	43	41	100%
Triang.	.1048	.1073	.1126	.1163	.1222	.1259	
Quad.	.1036	.1064	.1108	.1136	.1177	.1194	

Table 9. Variances and Relative Efficiencies of the Non-Symmetrized Estimators $n = 10$ $N = 1000$

k =	3	4	5	6	7	8	BLUE
Normal	.2614	.3096	.3374	.3533	.3599	.3657	.1000
	38	32	30	28	28	27	100%
CN .05	.2795	.3285	.3584	.3728	.3758	.3766	.1198
	43	36	33	32	32	32	100%
CN .10	.3182	.3670	.3963	.4082	.4088	.4078	.1358
	43	37	34	33	33	33	100%
Logist	.2109	.2460	.2628	.2701	.2763	.2749	.0934
	44	38	36	35	34	34	100%
Dl. Exp.	.2764	.3066	.3222	.3318	.3382	.3340	.1399
	51	46	43	42	41	42	100%
Spec.	.2047	.2351	.2492	.2506	.2623	.2609	.0957
	47	41	38	38	36	37	100%
Cauchy	.4352	.3541	.3281	.3025	.3020	.3105	.1631
	37	46	50	54	54	53	100%
Rect.	.4829	.5984	.6586	.6877	.6912	.6871	.0455
	9	8	7	7	7	7	100%
Triang.	.3086	.3602	.3876	.4022	.4022	.4030	
Quad.	.2744	.3184	.3443	.3551	.3507	.3613	

Table 3. The Ratios of the Conditional Variances and the Variance of Estimators

$n = 10 \qquad c_k = \frac{(n-k)(n-k+1)}{k(n-1)}$

k =	3	4	5	6	7	8
Normal	2.017	1.076	0.670	0.334	0.183	0.0767
CN .05	2.051	1.224	0.699	0.404	0.192	0.0809
CN .10	2.066	1.240	0.717	0.416	0.199	0.0847
Logist	2.097	1.247	0.706	0.404	0.196	0.0847
Dl. Exp.	2.207	1.317	0.735	0.428	0.210	0.0905
Special	2.122	1.267	0.719	0.412	0.200	0.0869
Cauchy	2.136	1.675	1.205	0.674	0.289	0.1195
Rect.	2.178	1.274	0.683	0.372	0.161	0.0633
Triang.	2.007	1.170	0.641	0.362	0.169	0.0697
Quad.	2.073	1.219	0.681	0.390	0.185	0.0794
c_k	2.074	1.167	0.667	0.390	0.190	0.0833

k = 3

	N	CN.05	CN.10	L[illegible]	DE	Spec	Cauchy	Rect	Tri	Quad
-4.0	1	1	1	1	1	1	0	3	1	5
-3.8	4	1	1	1	1	1	1	4	2	6
-3.6	5	1	1	3	1	1	1	4	4	8
-3.4	7	3	1	3	1	3	1	5	5	9
-3.2	8	4	1	3	2	4	1	5	9	10
-3.0	11	7	4	3	2	4	2	7	12	13
-2.8	13	10	8	8	4	7	3	12	17	15
-2.6	15	18	14	10	6	9	3	17	25	18
-2.4	20	22	18	15	14	13	3	20	30	24
-2.2	29	29	23	22	24	23	5	23	36	28
-2.0	40	36	34	32	35	37	7	31	46	36
-1.8	52	48	48	48	50	51	18	42	65	50
-1.6	70	65	67	73	68	74	36	67	79	69
-1.4	102	97	94	112	91	103	59	103	104	94
-1.2	138	136	134	140	113	136	101	127	129	123
-1.0	170	170	169	187	155	173	146	171	173	164
-0.8	227	230	231	232	209	225	190	215	227	207
-0.6	289	301	304	301	280	287	271	271	287	274
-0.4	352	364	371	367	353	357	344	345	339	359
-0.2	435	445	449	429	422	431	423	418	414	446
0.0	508	509	520	494	491	504	492	484	477	513
0.2	590	588	591	568	561	572	575	576	559	595
0.4	648	658	653	642	633	632	659	667	637	671
0.6	710	714	717	717	710	701	733	740	713	730
0.8	771	773	773	776	762	771	789	793	782	784
1.0	822	828	828	824	826	817	862	847	832	832
1.2	862	863	866	866	871	861	901	875	863	859
1.4	907	901	901	898	904	888	935	913	899	889
1.6	933	931	931	925	926	926	960	936	924	917
1.8	951	952	951	948	942	954	976	949	935	937
2.0	960	964	965	961	964	970	982	962	949	951
2.2	968	970	973	977	979	982	985	972	968	960
2.4	978	980	981	984	986	988	991	979	973	970
2.6	984	985	988	990	990	990	996	986	980	979
2.8	988	990	991	993	995	994	996	990	986	984
3.0	991	992	993	994	998	997	997	992	990	990
3.2	994	993	994	996	999	997	1000	994	993	993
3.4	996	995	994	997	1000	997	1000	997	996	994
3.6	996	996	997	998	1000	999	1000	997	997	995
3.8	998	998	998	999	1000	999	1000	998	998	997
4.0	998	998	998	999	1000	999	1000	998	998	998

Table 4a. CDF of studentized estimator.

t d.f. 8. $t_{0.995} = 3.355$ $t_{0.99} = 2.896$ $t_{0.975} = 2.306$

$t_{0.95} = 1.860$ $t_{0.90} = 1.397$ $t_{0.75} = 0.706$

k = 4

	N	CN.05	CN.10	Lo	DE	Spec	Cauchy	Rect	Tri	Quad
-4.0	2	0	1	1	1	2	1	3	1	6
-3.8	5	2	1	1	1	2	1	3	3	6
-3.6	6	3	1	2	1	2	1	5	5	7
-3.4	8	4	1	3	2	3	1	5	6	8
-3.2	9	6	3	3	2	3	1	5	11	10
-3.0	10	7	4	3	2	4	1	6	13	12
-2.8	14	11	7	5	3	5	2	11	17	14
-2.6	16	16	11	9	8	7	3	15	22	16
-2.4	18	21	15	15	12	18	4	20	30	23
-2.2	27	27	23	19	24	24	5	24	37	26
-2.0	38	34	33	36	30	32	8	30	45	34
-1.8	49	46	42	54	48	51	16	52	59	51
-1.6	67	61	63	74	68	68	30	68	76	68
-1.4	98	91	87	104	87	101	51	86	101	95
-1.2	130	128	125	135	112	135	76	117	128	121
-1.0	167	168	163	177	145	172	134	156	168	157
-0.8	221	228	225	229	204	218	176	203	224	202
-0.6	289	297	296	294	268	282	254	257	286	280
-0.4	354	363	374	365	341	351	337	335	339	353
-0.2	428	438	439	431	422	420	413	417	406	440
0.0	508	513	522	499	491	502	489	489	475	516
0.2	584	580	581	578	567	575	579	584	563	590
0.4	653	660	654	646	636	633	667	675	636	658
0.6	711	719	721	717	715	708	738	752	717	737
0.8	770	772	774	782	779	772	808	809	779	787
1.0	827	834	833	829	836	827	867	854	833	836
1.2	869	869	874	863	872	867	905	881	873	848
1.4	920	905	906	896	910	900	942	919	900	892
1.6	935	933	932	929	925	928	965	937	924	915
1.8	951	956	955	949	949	948	977	946	939	938
2.0	959	964	966	965	966	970	982	964	951	954
2.2	968	969	972	974	977	982	987	971	964	960
2.4	979	981	984	981	983	989	992	979	972	974
2.6	983	986	989	991	990	991	994	986	978	981
2.8	987	990	991	993	995	996	997	990	985	986
3.0	993	993	993	996	995	997	999	992	991	990
3.2	995	994	994	997	998	997	999	997	991	993
3.4	997	996	996	997	999	998	999	997	995	994
3.6	998	997	997	998	999	999	999	997	996	995
3.8	998	998	998	998	1000	999	1000	997	997	996
4.0	998	998	999	998	1000	999	1000	997	999	998

Table 4b. CDF of studentized estimator.

t d.f. 7 $t_{0.995} = 3.499$ $t_{0.99} = 2.998$ $t_{0.975} = 2.365$

$t_{0.95} = 1.895$ $t_{0.90} = 1.415$ $t_{0.75} = 0.711$

k = 5

	N	CN.05	CN.10	LO	DE	Sper	Cauchy	Rect	Tri	Quad
-4.0	6	3	3	3	3	2	1	3	3	6
-3.8	8	4	3	3	3	4	3	5	5	7
-3.6	9	6	5	3	3	4	3	6	7	9
-3.4	9	6	5	4	4	4	3	8	8	10
-3.2	9	7	6	5	5	6	3	10	12	10
-3.0	12	10	8	7	6	7	3	13	15	14
-2.8	16	14	9	10	7	11	4	14	19	18
-2.6	18	18	15	14	11	13	6	17	22	20
-2.4	20	23	17	19	15	21	10	22	31	23
-2.2	34	33	32	25	26	33	12	34	43	30
-2.0	43	42	40	42	35	40	15	41	52	34
-1.8	51	50	49	59	51	57	23	59	64	58
-1.6	70	65	70	72	65	79	39	71	82	73
-1.4	100	92	89	104	86	104	59	87	104	100
-1.2	131	124	121	143	111	135	95	118	137	129
-1.0	177	173	167	179	146	172	140	158	179	164
-0.8	226	228	222	230	211	222	190	202	229	204
-0.6	228	297	295	287	269	284	252	264	284	279
-0.4	355	359	367	364	337	358	322	327	349	347
-0.2	421	431	433	431	418	419	407	415	403	435
0.0	506	510	515	500	497	497	493	496	479	514
0.2	579	580	586	581	568	571	585	587	566	588
0.4	649	650	645	645	639	638	609	676	634	655
0.6	711	712	715	715	707	717	737	756	712	728
0.8	765	767	772	774	788	772	810	815	771	785
1.0	822	825	820	819	827	820	855	850	823	829
1.2	863	860	864	861	875	859	896	880	863	859
1.4	901	896	895	892	905	898	940	910	903	887
1.6	928	925	931	920	924	930	957	924	915	911
1.8	939	948	953	943	945	946	971	939	930	930
2.0	953	957	960	956	966	963	981	953	948	943
2.2	965	966	969	967	972	977	984	961	958	960
2.4	972	974	975	977	980	982	988	971	969	970
2.6	977	979	978	985	985	989	991	976	976	975
2.8	983	986	986	989	992	991	992	985	979	978
3.0	988	989	989	992	994	994	996	989	985	986
3.2	993	993	993	994	995	996	998	990	989	989
3.4	996	995	995	995	996	997	998	992	991	989
3.6	998	997	996	996	996	997	998	995	991	990
3.8	998	997	997	996	997	997	999	995	991	991
4.0	998	998	998	997	998	997	999	995	993	993

Table 4c. CDF of studentized estimator.

t d.f. 6 $t_{0.995} = 3.706$ $t_{0.99} = 3.143$ $t_{0.975} = 2.447$

$t_{0.95} = 1.943$ $t_{0.90} = 1.440$ $t_{0.75} = 0.718$

k = 6

	N	CN.05	CN.10	Lo	DE	Spec	Cauchy	Rect	Tri	Quad
-5.0	1	1	2	2	3	2	0	2	1	3
-4.8	1	1	3	2	3	2	0	2	1	4
-4.6	2	1	3	2	3	2	0	2	2	5
-4.4	3	2	3	2	3	3	1	2	2	5
-4.2	4	3	3	2	3	3	1	2	3	5
-4.0	5	3	3	2	3	3	1	6	5	6
-3.8	7	5	5	3	3	3	2	6	6	8
-3.6	9	6	5	4	3	5	2	7	7	9
-3.4	9	6	5	4	4	6	3	9	8	11
-3.2	10	7	6	5	6	9	3	14	9	12
-3.0	10	10	10	10	8	12	5	14	15	14
-2.8	17	16	11	12	8	12	5	16	20	18
-2.6	20	19	16	16	13	15	7	18	26	21
-2.4	26	26	22	16	21	21	9	24	30	24
-2.2	32	33	32	25	27	31	12	35	46	34
-2.0	43	42	42	42	41	43	17	46	57	43
-1.8	58	55	55	60	51	56	29	59	70	56
-1.6	72	65	66	73	60	80	41	75	88	72
-1.4	98	87	88	107	81	104	61	93	107	97
-1.2	138	128	124	141	114	133	94	125	137	129
-1.0	179	169	163	182	144	182	135	156	175	164
-0.8	221	223	215	229	206	223	178	201	225	213
-0.6	287	287	288	288	264	285	247	270	284	276
-0.4	357	361	366	360	335	353	310	326	339	341
-0.2	419	430	436	425	417	421	408	408	407	425
0.0	507	510	514	503	506	496	492	495	480	511
0.2	579	578	582	576	577	575	589	587	565	593
0.4	649	650	646	650	639	643	673	675	641	655
0.6	713	715	723	717	715	716	741	758	712	727
0.8	766	770	774	781	788	778	804	818	771	785
1.0	819	824	824	816	827	818	849	851	817	823
1.2	865	867	870	864	874	863	904	881	867	863
1.4	894	894	899	892	965	894	935	903	900	889
1.6	920	920	923	922	927	929	951	919	919	910
1.8	939	940	942	942	952	943	968	932	930	928
2.0	948	951	955	954	963	963	982	949	945	945
2.2	964	966	966	966	972	976	984	959	959	958
2.4	970	972	973	976	978	984	988	966	968	966
2.6	974	974	974	985	984	985	991	975	975	975
2.8	982	984	984	988	989	990	992	982	981	981
3.0	983	986	987	988	990	994	993	987	982	985
3.2	992	991	992	990	996	995	994	987	987	985
3.4	995	995	993	993	996	996	996	989	988	987
3.6	997	996	996	995	996	996	997	990	990	988
3.8	998	997	997	996	997	997	998	994	992	989
4.0	998	998	998	997	997	998	998	995	992	993
4.2	999	999	999	997	997	998	999	996	993	994
4.4	999	999	1000	997	998	998	999	996	993	996
4.6	999	999	1000	997	998	998	1000	996	993	996
4.8	999	999	1000	998	999	998	1000	996	996	996
5.0	999	999	1000	998	999	999	1000	996	996	998

Table 4d. CDF of studentized estimator.

t d.f. 5 $t_{0.995} = 4.032$ $t_{0.99} = 3.365$ $t_{0.975} = 2.571$
$t_{0.95} = 2.015$ $t_{0.90} = 1.440$ $t_{0.75} = 0.727$

k = 7

	N	CN.05	CN.10	Lo	DE	Spec	Cauchy	Rect	Tri	Quad
-5.0	5	5	4	10	5	4	4	11	5	11
-4.8	5	5	4	10	5	4	6	12	6	12
-4.6	5	5	5	10	5	4	7	12	8	13
-4.4	5	6	6	11	6	5	9	13	10	15
-4.2	9	7	6	11	6	5	10	13	10	15
-4.0	10	8	8	12	6	7	10	15	12	16
-3.8	13	11	11	13	6	9	10	16	15	17
-3.6	18	14	14	14	7	10	10	19	16	19
-3.4	20	16	16	16	8	11	13	21	18	21
-3.2	22	19	18	18	10	15	16	25	22	23
-3.0	23	22	22	21	11	18	16	27	26	27
-2.8	27	26	26	23	16	22	17	30	32	31
-2.6	32	33	31	29	23	30	18	35	41	41
-2.4	35	35	35	35	26	39	24	46	49	48
-2.2	44	45	44	44	36	49	28	59	60	56
-2.0	53	58	58	56	47	53	35	68	72	59
-1.8	77	71	71	79	61	76	43	84	87	74
-1.6	95	56	56	96	76	92	59	100	105	94
-1.4	120	112	110	123	97	120	74	115	127	122
-1.2	156	149	145	156	122	156	109	143	154	153
-1.0	189	180	179	200	157	201	145	181	195	186
-0.8	232	227	228	241	211	230	192	221	238	240
-0.6	291	294	297	297	274	301	260	278	274	288
-0.4	362	365	365	362	341	363	321	337	352	354
-0.2	426	433	440	425	421	424	407	414	409	421
0.0	499	501	507	502	502	498	497	489	492	510
0.2	568	574	572	576	575	572	581	577	587	581
0.4	638	644	643	637	641	637	666	656	635	649
0.6	701	699	707	699	706	705	736	739	694	709
0.8	753	758	760	768	771	766	790	793	752	778
1.0	793	797	805	801	810	801	838	831	812	810
1.2	841	841	847	841	848	848	883	860	851	843
1.4	883	881	879	877	883	881	912	894	840	876
1.6	902	908	914	909	905	906	932	901	899	891
1.8	922	922	927	922	927	922	943	914	914	909
2.0	938	935	938	936	949	950	961	927	933	925
2.2	948	949	947	950	962	960	972	940	947	941
2.4	956	958	956	960	966	965	979	955	955	951
2.6	961	961	962	966	972	974	985	962	963	957
2.8	966	968	971	977	978	979	987	970	969	964
3.0	975	974	976	981	983	982	989	974	976	968
3.2	981	979	981	984	987	983	989	977	976	973
3.4	983	982	983	988	990	984	991	981	978	976
3.6	986	984	986	990	991	987	992	986	980	979
3.8	988	987	989	991	994	987	994	987	983	981
4.0	992	990	991	993	994	989	995	989	987	985
4.2	992	991	992	994	995	989	995	991	989	989
4.4	993	994	997	995	995	989	995	992	989	991
4.6	995	996	999	996	995	990	998	993	991	994
4.8	996	997	999	996	997	990	998	995	993	994
5.0	997	997	999	996	997	990	998	996	994	996

Table 4e. CDF of studentized estimator.

t d.f. 4 $t_{0.995} = 4.604$ $t_{0.99} = 3.747$ $t_{0.975} = 2.776$

$t_{0.95} = 2.132$ $t_{0.90} = 1.533$ $t_{0.75} = 0.741$

k = 8

	N	CN.05	CN.10	Lo	DE	Spec	Cauchy	Rect	Tri	Quad
-5.0	8	7	8	11	6	4	9	16	8	12
-4.8	9	7	9	11	6	4	11	19	10	13
-4.6	9	7	9	12	7	4	11	20	11	18
-4.4	12	9	11	14	7	7	11	21	13	19
-4.2	13	10	13	16	7	9	11	21	16	20
-4.0	16	10	13	18	8	13	12	21	16	20
-3.8	17	12	16	19	8	15	12	23	18	23
-3.6	20	16	20	21	10	18	14	26	21	25
-3.4	24	23	27	23	13	23	15	27	24	29
-3.2	27	27	29	25	20	28	15	30	28	31
-3.0	30	29	31	28	21	36	17	35	33	37
-2.8	34	32	34	31	32	41	22	43	43	43
-2.6	38	35	38	41	38	48	24	56	54	47
-2.4	49	45	44	46	44	53	34	65	68	51
-2.2	60	54	53	53	51	62	38	78	79	65
-2.0	74	64	66	72	63	73	45	90	90	74
-1.8	100	90	85	93	72	84	57	103	102	87
-1.6	120	113	108	119	95	113	71	120	119	104
-1.4	133	124	121	140	110	139	93	141	145	127
-1.2	164	157	154	169	135	170	125	173	175	161
-1.0	201	193	192	213	174	202	167	216	214	195
-0.8	250	238	237	256	222	251	214	243	257	242
-0.6	301	299	306	305	289	302	267	289	306	288
-0.4	365	372	374	368	346	369	334	349	367	361
-0.2	437	446	447	438	430	425	408	422	420	425
0.0	497	505	510	505	506	497	500	498	485	501
0.2	565	571	571	575	581	563	580	563	559	578
0.4	633	642	641	630	639	635	669	639	630	651
0.6	695	695	695	689	704	697	726	720	684	702
0.8	742	750	747	749	763	748	781	770	743	766
1.0	782	778	785	795	805	793	832	813	796	801
1.2	825	825	833	827	845	837	868	840	833	834
1.4	858	860	871	855	870	865	892	863	861	858
1.6	888	886	900	879	897	883	917	881	883	880
1.8	907	905	915	909	916	912	927	897	895	894
2.0	924	924	927	925	935	936	944	912	914	911
2.2	936	935	937	935	948	947	957	926	925	926
2.4	946	944	946	944	959	955	965	939	938	941
2.6	950	950	949	951	964	964	971	948	949	948
2.8	956	957	955	957	971	970	978	953	957	952
3.0	963	960	961	961	975	973	980	957	963	955
3.2	967	963	963	970	979	977	983	961	967	959
3.4	970	968	967	978	985	979	985	970	969	964
3.6	973	970	972	981	989	979	986	978	971	967
3.8	975	971	974	984	990	982	988	981	974	973
4.0	979	976	979	987	991	984	990	982	976	977
4.2	982	979	983	990	994	985	993	985	980	979
4.4	984	981	985	994	996	989	995	988	983	981
4.6	986	984	989	994	997	989	995	991	986	985
4.8	987	987	992	995	997	989	995	992	986	987
5.0	989	990	994	996	997	990	995	992	990	989

Table 4f. CDF of studentized estimator.

t d.f. 3 $t_{0.995} = 5.841$ $t_{0.99} = 4.541$ $t_{0.975} = 3.182$
$t_{0.95} = 2.353$ $t_{0.90} = 1.638$ $t_{0.75} = 0.756$

Table 5. Variances and Efficiencies of Estimators

k = 0	n =	10	15	20
	N =	1000	1000	1000
Normal	Variance	.1157	.0757	.0532
	V of BLUE	.1000	.0657	.0496
	Efficiency	87%	90%	94%
CN .05	V	.1309	.0839	.0621
	BLUE	.1199	.0786	.0585
	Eff	92	94	96
CN .10	V	.1524	.0956	.0699
	BLUE	.1358	.0886	.0657
	Eff	89	92	94
Logist	V	.1016	.0659	.0452
	BLUE	.0934	.0618	.0462
	Eff	92	93	100
Dl. Exp.	V	.1626	.1039	.0737
	BLUE	.1399	.0880	.0637
	Eff.	86	85	86
Special	V	.1078	.0692	.0472
	BLUE	.0966		
	Eff.	89		
Cauchy	V	.2220	.1281	.0805
	BLUE	.1631	.0911	.0628
	Eff	74	71	78
Rect.	V	.0953	.0503	.0316
	BLUE	.0455	.0221	.0130
	Eff.	47	44	42

Table 6A. Estimated Variances and Efficiences of the Estimators by the "Difference" Estimation n =10
M=Mean V=Variance CI=95% Var=Variance Eff.in %

	k =	3		4		5	
	N =	2000		2000		2000	
		Var.	Eff.	Var.	Eff.	Var.	Eff.
N	M	.1017	98.3	.1037	96.4	.1069	93.5
	V	.0008	97.1	.0017	94.8	.0030	91.5
	CI	±.0013	~99.6	±.0018	~98.1	±.0024	~95.7
CN .05	M	.1225	97.6	.1226	97.6	.1226	97.6
	V	.0029	95.8	.0033	95.5	.0055	95.0
	CI	±.0024	~99.6	±.0026	~99.7	±.0033	~100.0
CN .10	M	.1466	92.6	.1457	93.2	.1437	94.5
	V	.0089	90.1	.0077	90.8	.0074	92.1
	CI	±.0042	~95.4	±.0039	~95.8	±.0038	~97.1
Logist	M	.0964	96.9	.0982	95.2	.1000	93.5
	V	.0015	95.9	.0023	93.2	.0030	91.3
	CI	±.0012	~98.2	±.0021	~97.2	±.0024	~95.7
Dl.Exp.	M	.1706	82.0	.1703	82.1	.1677	83.4
	V	.0236	78.9	.0252	78.9	.0256	80.0
	CI	±.0068	~85.4	±.0070	~85.7	±.0071	~87.1
Special	M	.1041	91.9	.1051	91.1	.1053	90.9
	V	.0044	89.4	.0052	88.3	.0057	88.0
	CI	±.0030	~94.7	±.0033	~94.0	±.0034	~93.9
Cauchy	M	.9042	18.0	.6478	25.2	.2368	68.9
	V	∞		∞		.2296	62.6
	CI					±.0239	~76.6
Rect.	M	.0957	47.5	.0915	49.7	.0948	47.9
	V	.0008	46.9	.0021	48.6	.0049	46.4
	CI	±.0012	~.48.1	±.0020	~50.8	±.0031	~49.6

Table 6b. n = 10
Eff in %

	k =	6		7		8	
	N =	4000		4000		4000	
		Var.	Eff.	Var.	Eff.	Var.	Eff.
N	M	.1107	90.3	.1153	86.7	.1181	84.7
	V	.0049	88.6	.0063	84.9	.0071	82.8
	CI	±.0022	~92.2	±.0025	~88.7	±.0027	~86.7
CN .05	M	.1258	95.1	.1311	91.2	.1330	89.9
	V	.0070	93.1	.0081	89.3	.0083	88.0
	CI	±.0026	~97.1	±.0028	~93.2	±.0029	~92.0
CN .10	M	.1438	94.4	.1491	91.1	.1517	90.1
	V	.0088	92.6	.0105	89.2	.0111	88.2
	CI	±.0029	~96.4	±.0032	~93.1	±.0033	~92.1
Logist	M	.1014	92.2	.1043	89.6	.1050	89.0
	V	.0036	90.5	.0049	87.7	.0060	87.0
	CI	±.0019	~93.9	±.0022	~91.5	±.0024	~91.1
Dl. Exp.	M	.1699	82.3	.1682	83.2	.1706	81.7
	V	.0298	79.8	.0260	80.7	.0307	79.0
	CI	±.0055	~85.1	±.0051	~85.8	±.0056	~84.4
Special	M	.1060	90.3	.1080	88.7	.1086	88.1
	V	.0058	88.3	.0071	86.7	.0088	85.8
	CI	±.0024	~92.4	±.0026	~90.8	±.0030	~90.6
Cauchy	M	.2393	68.2	.2873	56.8	.3499	46.6
	V	.2913	63.6	∞		∞	
	CI	±.0171	~73.4				
Rect.	M	.0961	47.8	.1061	42.8	.1087	41.8
	V	.0217	45.5	.0090	41.7	.0092	40.7
	CI	±.0047	~50.2	±.0030	~44.1	±.0030	~43.0

Table 7A. Estimated Variances and Efficiencies of the Estimators by the "Difference" Estimation
n = 15 M=Mean Var=Variance C.I.=95%
Efficiency is given in percentages.

	k =	3		4		5	
	N =	2000		2000		2000	
		Var.	Eff.	Var.	Eff.	Var.	Eff.
N	M	.0678	98.4	.0692	96.4	.0708	94.2
	V	.0002	97.5	.0005	95.0	.0008	92.5
	CI	±.0006	~99.3	±.0010	~97.8	±.0013	~96.0
CN..05	M	.0818	96.1	.0814	96.5	.0819	96.0
	V	.0012	94.3	.0012	94.7	.0016	93.9
	CI	±.0016	~98.0	±.0016	~98.5	±.0018	~98.1
CN .10	M	.0977	90.7	.0956	92.7	.0948	93.5
	V	.0035	88.3	.0027	90.5	.0032	91.1
	CI	±.0026	~93.2	±.0023	~95.0	±.0025	~96.0
Logist	M	.0637	97.0	.0643	96.1	.0651	94.9
	V	.0007	95.2	.0008	94.4	.0009	93.1
	CI	±.0012	~98.9	±.0012	~97.9	±.0013	~96.9
Dl. Exp.	M	.1118	78.7	.1097	80.2	.1077	81.7
	V	.0100	75.7	.0094	77.1	.0088	78.6
	CI	±.0045	~82.0	±.0044	~83.5	±.0042	~85.0
Cauchy	M	.5124	17.8	.2876	31.7	.1484	61.4
	V	∞		∞		.1069	55.7
	CI					±.0150	~68.3
Rect.	M	.0599	36.9	.0545	40.6	.0516	42.8
	v	.0002	36.5	.0004	39.7	.0012	41.5
	CI	±.0006	~37.3	±.0009	~41.5	±.0016	~44.2

Figure 7B. n = 15
Eff. in %

	k =	6		7		8	
	N =	4000		4000		4000	
		Var.	Eff.	Var.	Eff.	Var.	Eff.
N	M	.0721	92.5	.0726	91.9	.0730	91.4
	V	.0013	91.0	.0017	90.3	.0020	89.7
	CI	±.0012	~94.1	±.0013	~93.5	±.0014	~93.2
CN .05	M	.0819	96.0	.0840	93.6	.0843	93.2
	V	.0020	94.4	.0025	91.8	.0027	91.5
	CI	±.0014	~97.6	±.0016	~95.4	±.0016	~95.0
CN .10	M	.0934	94.9	.0957	92.6	.0956	92.7
	V	.0028	93.2	.0037	90.8	.0040	90.8
	CI	±.0017	~96.6	±.0019	~94.5	±.0020	~94.7
Logist	M	.0651	94.9	.0665	92.9	.0681	90.7
	V	.0011	93.5	.0014	91.3	.0018	89.0
	CI	±.0010	~96.4	±.0012	~94.6	±.0013	~92.5
Dl. Exp.	M	.1079	81.6	.1071	82.2	.1079	81.6
	V	.0099	79.2	.0105	79.7	.0117	78.6
	CI	±.0032	~84.0	±.0032	~84.7	±.0034	~84.7
Cauchy	M	.1299	70.1	.1128	80.8	.1100	82.8
	V	.0405	66.9	.0304	77.1	.0392	78.2
	CI	±.0064	~73.8	±.0055	~85.0	±.0063	~87.9
Rect.	M	.0496	44.6	.0506	43.5	.0510	43.3
	V	.0058	42.5	.0022	42.4	.0028	41.9
	CI	±.0024	~46.8	±.0015	~44.8	±.0017	~44.8

Table 8A. Estimated Variances and Efficiencies of the Estimators by the "Difference" Estimation. n = 20 M=Mean Var=Variance CI,95% Eff. in %

	k =	3		4		5	
	N =	2000		2000		2000	
		Var.	Eff.	Var.	Eff.	Var.	Eff.
N	M	.0506	98.8	.0513	97.4	.0521	96.0
	V	.0001	98.0	.0002	96.2	.0003	94.5
	CI	±.0004	~99.6	±.0006	~98.6	±.0008	~97.5
CN .05	M	.0610	95.9	.0606	96.5	.0606	96.5
	V	.0008	94.1	.0007	94.6	.0007	94.6
	CI	±.0012	~97.8	±.0012	~98.6	±.0012	~98.6
CN .10	M	.0715	92.0	.0696	94.5	.0686	95.8
	V	.0019	89.5	.0013	92.3	.0012	93.7
	CI	±.0020	~94.6	±.0016	~96.7	±.0016	~98.0
Logist	M	.0480	96.3	.0481	96.1	.0484	95.5
	V	.0004	94.6	.0004	94.4	.0004	93.7
	CI	±.0009	~98.1	±.0009	~97.9	±.0009	~97.3
Dl. Exp.	M	.0839	75.9	.0809	78.7	.0785	81.1
	V	.0067	71.6	.0058	75.6	.0058	77.8
	CI	±.0036	~79.2	±.0034	~82.2	±.0034	~84.8
Cauchy	M	.2107	29.8	.1435	43.7	.0949	66.2
	V					.0142	62.6
	CI					±.0054	~70.2
Rect.	M	.0442	29.2	.0395	32.9	.0364	35.7
	V	.0001	28.9	.0003	32.2	.0006	34.7
	CI	±.0004	~29.7	±.0008	~33.6	±.0011	~36.8

Table 5B. n = 20

Eff. in %

	k =	6		7		8	
	N =	4000		4000		4000	
		Var.	Eff.	Var.	Eff.	Var.	Eff.
N	M	.0527	94.9	.0536	93.4	.0543	92.1
	V	.0005	93.7	.0006	92.0	.0007	90.6
	CI	±.0007	~96.1	±.0008	~94.8	±.0009	~93.6
CN .05	M	.0607	96.4	.0619	94.5	.0625	93.6
	V	.0008	95.0	.0011	93.0	.0012	92.0
	CI	±.0009	~97.8	±.0010	~96.1	±.0011	~95.3
CN .10	M	.0690	95.4	.0699	94.1	.0703	93.6
	V	.0012	93.9	.0013	92.7	.0019	91.8
	CI	±.0011	~97.0	±.0011	~95.6	±.0014	~95.5
Logist	M	.0483	95.7	.0499	92.6	.0504	91.7
	V	.0005	94.3	.0007	91.1	.0008	90.1
	CI	±.0007	~97.1	±.0008	~94.1	±.0009	~93.3
Dl. Exp.	M	.0783	81.3	.0768	83.0	.0768	83.0
	V	.0056	78.9	.0055	80.4	.0057	80.4
	CI	±.0024	~83.9	±.0024	~85.6	±.0024	~85.6
Cauchy	M	.0857	73.3	.0802	78.3	.0779	80.6
	V	.0113	70.5	.0094	75.4	.0087	77.7
	CI	±.0034	~76.3	±.0031	~81.4	±.0029	~83.7
Rect.	M	.0336	38.7	.0334	38.9	.0326	39.9
	V	.0031	36.7	.0011	37.8	.0013	38.6
	CI	±.0018	~40.9	±.0010	~40.1	±.0011	~41.3

k =	level	3	4	5	6	7	8
Normal	99 %	19	18	16	15	20	30
	98	40	30	32	26	44	61
	95	84	77	87	89	112	134
	90	189	171	187	183	119	237
CN.05	99 %	16	14	14	18	19	30
	98	31	28	28	27	40	56
	95	81	76	78	86	103	135
	90	188	156	168	172	196	225
CN.10	99 %	12	11	13	18	17	30
	98	28	24	27	25	39	52
	95	75	57	74	81	102	134
	90	167	147	166	169	200	214
Logist	99 %	15	12	17	15	28	18
	98	28	20	30	31	52	36
	95	91	82	87	72	103	92
	90	190	167	177	168	199	202
Dl.Exp.	99 %	5	6	11	9	17	15
	98	29	21	22	22	35	28
	95	79	66	83	63	86	82
	90	174	173	162	136	178	174
Special	99 %	15	12	17	15	28	18
	98	26	19	28	23	49	35
	95	84	72	76	60	99	91
	90	181	159	174	167	190	198
t d.f.2	99 %	12	8	10	12	25	15
	98	20	15	23	17	43	30
	95	59	53	63	58	88	76
	90	143	125	127	143	167	174
Cauchy	99 %	3	4	6	8	17	12
	98	11	8	11	11	32	25
	95	18	32	47	33	65	53
	90	87	73	84	95	127	136
Rect	99 %	28	23	31	31	44	29
	98	45	41	54	56	71	56
	95	98	80	109	96	131	132
	90	186	178	192	175	247	258
Triang	99 %	24	18	24	20	31	24
	98	52	44	39	28	60	50
	95	120	118	106	83	124	126
	90	228	197	208	163	226	243
Quad	99 %	27	26	29	19	35	23
	98	55	24	41	29	64	57
	95	121	107	109	89	136	125
	90	210	198	210	183	241	235

Table 9. Error frequencies of confidence intervals.

n = 10 N = 2000

k =	level	3	4	5	6	7	8
Normal	99 %	20	17	18	15	22	16
	98	40	37	41	33	35	32
	95	103	92	96	97	94	94
	90	187	195	196	199	188	178
CN.05	99 %	15	14	16	14	19	18
	98	35	33	35	31	32	31
	95	88	85	85	87	80	86
	90	178	175	177	181	173	165
CN.10	99 %	18	15	17	14	17	12
	98	35	33	37	30	33	24
	95	86	76	77	80	78	61
	90	173	163	160	173	169	136
Logist	99 %	18	16	15	14	15	17
	98	30	31	35	36	30	34
	95	78	74	73	82	67	75
	90	186	169	166	154	168	169
Dl.Exp.	99 %	14	10	9	10	9	10
	98	25	19	16	25	16	22
	95	71	67	69	58	65	62
	90	169	159	154	137	155	146
Special	99 %	14	14	13	13	14	12
	98	36	32	32	33	29	29
	95	75	68	67	78	63	67
	90	180	164	162	146	160	154
t d.f.2	99 %	6	6	6	9	9	15
	98	29	20	21	26	20	22
	95	43	35	58	62	49	52
	90	151	133	131	118	133	124
Cauchy	99 %	3	2	2	6	2	4
	98	9	6	8	10	10	6
	95	32	30	33	34	30	29
	90	83	80	76	89	77	73
Rect	99 %	20	17	18	20	19	26
	98	40	32	34	40	31	44
	95	78	67	67	82	61	79
	90	160	137	140	147	128	155
Triang	99 %	28	27	26	26	24	20
	98	46	39	40	41	38	34
	95	91	85	84	91	90	92
	90	191	182	180	163	177	176
Quad	99 %	28	28	25	20	24	18
	98	49	43	40	40	35	34
	95	106	105	104	99	96	91
	90	201	193	194	186	186	160

Table 10. Error frequencies of confidence intervals.

n = 15 N = 2000

k =	level	3	4	5	6	7	8
Normal	99 %	14	19	18	14	20	19
	98	37	35	32	29	31	31
	95	89	84	86	95	85	89
	90	192	186	188	181	180	176
CN.05	99 %	13	11	13	15	17	17
	98	25	25	25	33	27	26
	95	85	85	89	84	83	87
	90	173	177	177	175	166	155
CN.10	99 %	13	11	10	15	13	13
	98	29	27	26	33	26	28
	95	80	78	84	85	80	80
	90	173	186	170	163	161	160
Logist	99 %	15	12	12	19	12	12
	98	35	35	29	32	29	27
	95	92	89	91	76	82	73
	90	173	167	168	181	176	175
Dl.Exp.	99 %	9	8	9	11	10	11
	98	17	18	19	29	19	20
	95	86	83	48	68	74	76
	90	184	175	170	147	165	154
Special	99 %	15	11	11	17	10	10
	98	31	27	27	27	28	27
	95	85	89	88	76	77	82
	90	168	161	165	170	166	161
t d.f.2	99 %	9	8	8	8	6	6
	98	17	20	21	18	22	21
	95	55	55	60	58	58	57
	90	139	129	129	134	119	131
Cauchy	99 %	5	2	2	1	2	2
	98	10	7	8	5	5	5
	95	17	22	29	23	33	35
	90	57	58	67	77	67	73
Rect	99 %	21	15	14	15	14	14
	98	34	31	30	24	27	31
	95	88	74	75	65	75	68
	90	154	144	135	135	121	124
Triang	99 %	28	25	24	19	23	21
	98	45	45	44	38	44	42
	95	97	87	90	84	90	92
	90	181	176	171	180	163	159
Quad	99 %	23	22	23	16	20	20
	98	47	44	48	40	46	47
	95	109	108	114	93	100	107
	90	209	198	195	191	202	202

Table 11. Error frequencies of confidence intervals.

n = 20 N = 2000

Table 12. Variances and Efficiencies of Estimators With Non-Negative Coefficients.

n =	5	5	5	10	10
N =	2000	2000	2000	2000	2000
k =	3	4	BLUE	3	4
Normal	.2121	.2177	.2000	.1016	.1035
	94.3	91.9	100%	98.4	96.6
CN .05	.2482	.2527	.2497	.1225	.1225
	100	98.8	100	97.6	97.6
CN .10	.2975	.3024	.2885	.1467	.1458
	97.0	95.4	100	92.6	93.2
Logist	.2005	.2088	.1910	.0963	.0978
	95.2	91.5	100	96.9	95.4
Dl. Exp.	.3496	.3641	.3169	.1707	.1705
	90.6	87.0	100	81.9	82.0
Spec.	.2124	.2218	.2021	.1041	.1050
	95.1	91.1	100	91.9	91.0
Cauchy	2.400	2.531	1.221	.9451	.7235
	50.9	48.2	100	17.2	22.2
Rect.	.2447	.2452	.1428	.0960	.0918
	58.3	58.2	100	47.8	49.5

§4. Observations on Monte Carlo Results.

The following observations are made on Monte Carlo results stated above.

1. Relative efficiencies with respect to BLUE are quite high for small sample sizes n = 10, 15, 20, and even for n =5.

2. Except for the double exponential, Cauchy and rectangular, all of which have some irregularities to be discussed in the following, the relative efficienties are uniformly higher than 90 percent, and quite often higher than 95 %.

3. Double exponential distribution has a discontinuity in the derivative of the density, hence one of the regularity conditions is not satisfied. Although the BLUE is asymptotically equivalent to the median and is fully efficient in this case, whether $\hat{\theta}^*$ is asymptotically fully efficient is open to question. The answer might be affirmative, but it cannot be derived from the combinations of general statements mentioned in §2.

4. The Cauchy distribution has no moments, and for sample size 3 or 4, no unbiased estimator with finite variance exists. This seems to give an intuitively natural explanation for the low efficiencies for the cases k = 3 and 4. Also in such cases the sample variances of $\hat{\theta}^* - \hat{\theta}^2_{BLUE}$ are so big and unstable, that they may well suggest that $E(\hat{\theta}^{*4})$ is actually infinity; hence we denote the facts by putting ∞ in the table. Even for the case $k \geq 5$, it is a problem whether $\hat{\theta}^*$ actually converges in probability to $\hat{\theta}^0_f$,

because the nonexistence of the moments makes the general arguments inapplicable.

5. For rectangular distribution, the variance of the BLUE $(= (X_{max} + X_{min})/2)$ is proportional to n^{-2} not to n^{-1} as is usually the case, hence it is expected that

$$V(\theta^*) \sim \frac{k}{n} V(\theta_f^0) \propto \frac{1}{nk} \quad ;$$

hence the relative efficiency will be approximately k/n.

6. Relative efficiencies with various k values depend on the shapes of the distribution. For the normal and also for sufficiently regular distributions such as the 5 percent contaminated normal and the logistic, smaller k gives higher relative efficiencies. For longer-tailed distributions larger k is better, although too large k gives uniformly poor efficiency.

7. The pattern of relative efficiencies with different values of k for each distribution is fairly stable as n varies from 10 to 20. For most cases the relative efficiency for fixed k increases, but not always, and even when it does, the increase is not steady or rapid.

8. The gain in the efficiency by symmetrization is surprising when n = 10. It may decrease as n increases, but it is not yet investigated. On the other hand the effect of restricting to non-negative coefficients is negligible when n = 10, which fact can be seen by comparing Table 6 and Table 12.

9. Confidence intervals based on the "studentization" described above show fairly uniform error probabilities except for Cauchy and the t-distribution with d.f. 2. Also, the error frequencies are nearly uniformly smaller than the prescribed level, which implies that a better approximation to the distribution should still be found.

10. If we take the normal case as the standard, confidence intervals tend to be biased more or less on the conservative side when the distribution is long-tailed, and seem to be slightly biased in the opposite direction when it is short-tailed.

Some further remarks are in order.

11. The variances of BLUE may not be a good criterion for calculating the relative efficiencies. The most proper bound will be the variance of Pitman's estimator which gives the sharp lower bound for all location invariant estimators, but the numerical values are not available. The Monte Carlo study now under way by V. Mikè (personal communication) indicates that the variance of the BLUE's is not very far from that of Pitman, and generally much closer to his than the Cramér-Rao bound, when the discrepancies are large (e.g. for such cases as the double exponential and the Cauchy). Hence the relative efficiences measured by the variance ratios with BLUE may be a little higher than more plausible ones; we can expect them to be not too far from them and to be a

reasonable approximation.

12. The relative efficiencies themselves are sometimes misleading, because they should be judged also in reference to those of other simpler estimators. For example, relative efficiencies of the sample mean (which is actually equal to the symmetrized estimator with k = 2) should be taken into account. Table 13 gives the efficiencies of the sample mean. Reference should also be made to the small sample efficiencies of the trimmed mean (Gastwirth and Cohen [12]).

13. The relative efficiencies could also be compared with the maximum efficiencies of linear estimators (Birnbaum et al. [6]) or of the maximum (invariant) estimators (Miké [21]). The numerical results obtained by them suggest that there remains little room for improving the efficiencies of our estimator over many shapes of distributions. For example when n = 20, the maximum efficiencies of linear estimators given in [6] are

Cauchy versus Normal	83.5 %
Double exponential vs. Normal	91.5 %
10% contaminated normal vs. Normal	96.6 %
Cauchy vs. Rectangular	22.5 %
Normal vs. Rectangular	67.5 %

Those values are not much better than the values attained by our method of a favorable choice of k.

Table 13. Relative Efficiencies of the Sample Mean. (%)

n =	5	10	15	20
Normal	100.0	100.0	100.0	100.0
CN .05	88.1	85.4	84.2	83.6
CN .10	80.1	75.4	73.9	73.7
Logist	95.5	93.5	92.7	92.3
Dl. Exp.	79.2	70.0	66.0	63.7
Special	85.7	81.1	*	*
Cauchy	0	0	0	0
Rect.	71.4	45.5	33.0	2.60

14. The best choice of k for given n depends also on the possible shapes of distributions, but roughly the following will be a plausible range.

n = 5	k = 3
10	5
15	5 ~ 6
20	5 ~ 7

More roughly, k = 5 seems to be a reasonable choice for the range of sample sizes n = 10 ~ 20, and for relatively regular distributions such as the normal, contaminated normal, or the logistic, we can guarantee relative efficiencies of approximately

93 %	for	n = 10
94		15
95		20

or slightly higher.

15. The relative efficiencies of the confidence intervals based on studentization will be lower than those of estimators because of the small degrees of freedom of the approximate t-distribution, especially when n-k is small. But the exact relative efficiencies are hard to assess, because little has been known about the small sample shortest confidence intervals for the median of non-normal distributions. Further investigation of this problem is yet to be done.

16. The regularity conditions required to establish the asymptotic full efficiency of our estimator are not explicitly stated. Although it is quite possible to state them rigorously by putting the results of several previous papers together, they may be unnecessarily restrictive, and it seems to be desirable to find a possible unified set of regularity conditions which are not too restrictive.

17. The same idea as proposed here can be applied to obtain robust procedures in other problems about location parameters. Application to the testing about one-sample and two-sample location parameters is discussed in another paper by the author [25]. Also, the estimated coefficients of BLUE can be used to test the hypothesis about the shapes of distribution (e.g. normality) against symmetric alternatives, which will be discussed in another paper [26].

18. After this work was completed, the author obtained a new result which gives a theoretical foundation, and also some generalization, to the method proposed here, and which will be given in a paper [27] now in preparation.

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